

ACTION OF THE CREMONA GROUP ON FOLIATIONS ON $\mathbb{P}_{\mathbb{C}}^2$: SOME CURIOUS FACTS

by

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Abstract. — The Cremona group of birational transformations of $\mathbb{P}_{\mathbb{C}}^2$ acts on the space $\mathbb{F}(2)$ of holomorphic foliations on the complex projective plane. Since this action is not compatible with the natural graduation of $\mathbb{F}(2)$ by the degree, its description is complicated. The fixed points of the action are essentially described by Cantat-Favre in [3]. In that paper we are interested in problems of "aberration of the degree" that is pairs $(\phi, \mathcal{F}) \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2) \times \mathbb{F}(2)$ for which $\deg \phi^* \mathcal{F} < (\deg \mathcal{F} + 1) \deg \phi + \deg \phi - 2$, the generic degree of such pull-back. We introduce the notion of numerical invariance ($\deg \phi^* \mathcal{F} = \deg \mathcal{F}$) and relate it in small degrees to the existence of transversal structure for the considered foliations.

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1. Introduction

Let us consider on the complex projective plane $\mathbb{P}_{\mathbb{C}}^2$ a foliation \mathcal{F} of degree d and a birational map ϕ of degree k . If the pair (\mathcal{F}, ϕ) is generic then $\deg \phi^* \mathcal{F} = (d + 1)k + k - 2$. For example if \mathcal{F} and ϕ are both of degree 2, then $\phi^* \mathcal{F}$ is of degree 6. Nevertheless one has the following statement which says that "aberration of the degree" is not exceptional:

Theorem A. — *For any foliation \mathcal{F} of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$ there exists a quadratic birational map ψ of $\mathbb{P}_{\mathbb{C}}^2$ such that $\deg \psi^* \mathcal{F} \leq 3$.*

Holomorphic singular foliations on compact complex projective surfaces have been classified up to birational equivalence by Brunella, McQuillan and Mendes ([1]). Let \mathcal{F} be a holomorphic singular foliation on a compact complex projective surface S . Let $\text{Bir}(\mathcal{F})$ (resp. $\text{Aut}(\mathcal{F})$) denote the group of birational (resp. biholomorphic) maps of S that send leaf to leaf. If \mathcal{F} is of general type, then $\text{Bir}(\mathcal{F}) = \text{Aut}(\mathcal{F})$ is a finite group. In [3] Cantat and Favre classify the pairs (S, \mathcal{F}) for which $\text{Bir}(\mathcal{F})$ (resp. $\text{Aut}(\mathcal{F})$) is infinite; in the case of $\mathbb{P}_{\mathbb{C}}^2$ such foliations are given by closed rational 1-forms.

In this article we introduce a weaker notion: the numerical invariance. We consider on $\mathbb{P}_{\mathbb{C}}^2$ a pair (\mathcal{F}, ϕ) of a foliation \mathcal{F} of degree d and a birational map ϕ of degree $k \geq 2$. The foliation \mathcal{F} is **numerically invariant** under the action of ϕ if $\deg \phi^* \mathcal{F} = \deg \mathcal{F}$. We characterize such pairs (\mathcal{F}, ϕ) with $\deg \mathcal{F} = \deg \phi = 2$ which

is the first degree with deep (algebraic and dynamical) phenomena, both for foliations and birational maps. We prove that a numerically invariant foliation under the action of a generic quadratic map is special:

Theorem B. — *Let \mathcal{F} be a foliation of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$ numerically invariant under the action of a generic quadratic birational map of $\mathbb{P}_{\mathbb{C}}^2$. Then \mathcal{F} is transversely projective.*

In that statement generic means outside an hypersurface in the space Bir_2 of quadratic birational maps of $\mathbb{P}_{\mathbb{C}}^2$.

For any quadratic birational map ϕ of $\mathbb{P}_{\mathbb{C}}^2$ there exists at least one foliation of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$ numerically invariant under the action of ϕ and we give "normal forms" for such foliations. We don't know if the foliations numerically invariant under the action of a non-generic quadratic birational map have a special transversal structure. Problem: for any birational map ϕ of degree $d \geq 3$, does there exist a foliation numerically invariant under the action of ϕ ?

A foliation \mathcal{F} on $\mathbb{P}_{\mathbb{C}}^2$ is **primitive** if $\deg \mathcal{F} \leq \deg \phi^* \mathcal{F}$ for any birational map ϕ . Foliations of degree 0 and 1 are defined by a rational closed 1-form (it is a well-known fact, see for example [2]). Hence a non-primitive foliation of degree 2 is also defined by a closed 1-form that is a very special case of transversely projective foliations. Generically a foliation of degree 2 is primitive. The following problem seems relevant: classify in any degree the primitive foliations numerically invariant under the action of birational maps of degree ≥ 2 ; are such foliations transversely projective or is this situation specific to the degree 2? In this vein we get the following statement.

Theorem C. — *A foliation \mathcal{F} of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$ numerically invariant under the action of a generic cubic birational map of $\mathbb{P}_{\mathbb{C}}^2$ satisfies the following properties:*

- \mathcal{F} is given by a closed rational 1-form (Liouvillian integrability);
- \mathcal{F} is non-primitive.

Is it a general fact, i.e. if \mathcal{F} is numerically invariant under the action of ϕ and $\deg \phi \gg \deg \mathcal{F}$ is \mathcal{F} Liouvillian integrable?

The text is organized as follows: we first give some definitions, notations and properties of birational maps of $\mathbb{P}_{\mathbb{C}}^2$ and foliations on $\mathbb{P}_{\mathbb{C}}^2$. In §3 we give a proof of Theorem A; we focus on foliations of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$ that have at least two singular points and then on foliations of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$ with exactly one singular point. The section 4 is devoted to the description of foliations of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$ numerically invariant under the action of any quadratic birational map. This allows us to prove Theorem B. At the end of the paper, §5, we describe the foliations of degree 2 numerically invariant under some cubic birational maps of $\mathbb{P}_{\mathbb{C}}^2$ and establish Theorem C.

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2. Some definitions, notations and properties

2.1. About birational maps of $\mathbb{P}_{\mathbb{C}}^2$. — A **rational map** ϕ of $\mathbb{P}_{\mathbb{C}}^2$ is a "map" of the type

$$\phi: \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2, \quad (x : y : z) \dashrightarrow (\phi_0(x, y, z) : \phi_1(x, y, z) : \phi_2(x, y, z))$$

where the ϕ_i 's are homogeneous polynomials of the same degree and without common factor. The **degree** of ϕ is by definition the degree of the ϕ_i 's. A **birational map** ϕ of $\mathbb{P}_{\mathbb{C}}^2$ is a rational map having a rational

"inverse" ψ , i.e. $\phi \circ \psi = \psi \circ \phi = \text{id}$. The first examples are the birational maps of degree 1 which generate the group $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) = \text{PGL}(3; \mathbb{C})$. Let us give some examples of quadratic birational maps:

$$\sigma: (x : y : z) \dashrightarrow (yz : xz : xy), \quad \rho: (x : y : z) \dashrightarrow (xy : z^2 : yz), \quad \tau: (x : y : z) \dashrightarrow (x^2 : xy : y^2 - xz).$$

These three maps, which are involutions, play an important role in the description of the set of quadratic birational maps of $\mathbb{P}_{\mathbb{C}}^2$.

The birational maps of $\mathbb{P}_{\mathbb{C}}^2$ form a group denoted $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ and called **Cremona group**. If ϕ is an element of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ then $\mathcal{O}(\phi)$ is the orbit of ϕ under the action of $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) \times \text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$:

$$\mathcal{O}(\phi) = \{ \ell \phi \ell' \mid \ell, \ell' \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^2) \}.$$

A very old theorem, often called Noether Theorem, says that any element of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ can be written, up to the action of an automorphism of $\mathbb{P}_{\mathbb{C}}^2$, as a composition of quadratic birational maps ([4]). In [5, Chapters 1 & 6] the structure of the set Bir_d (resp. $\mathring{\text{Bir}}_d$) of birational maps of $\mathbb{P}_{\mathbb{C}}^2$ of degree $\leq d$ (resp. d) has been studied when $d = 2$ and $d = 3$.

Theorem 2.1 (Corollary 1.10, Theorem 1.31, [5]). — *One has the following decomposition*

$$\mathring{\text{Bir}}_2 = \mathcal{O}(\sigma) \cup \mathcal{O}(\rho) \cup \mathcal{O}(\tau).$$

Furthermore

$$\text{Bir}_2 = \overline{\mathcal{O}(\sigma)}$$

where $\overline{\mathcal{O}(\sigma)}$ denotes the ordinary closure of $\mathcal{O}(\sigma)$, and

$$\dim \mathcal{O}(\tau) = 12, \quad \dim \mathcal{O}(\rho) = 13, \quad \dim \mathcal{O}(\sigma) = 14.$$

Note that there is a more precise description of Bir_2 in [5, Chapter 1].

We will further do some computations with birational maps of degree 3. Let us consider the following family of cubic birational maps:

$$\Phi_{a,b}: (x : y : z) \dashrightarrow (x(x^2 + y^2 + axy + bxz + yz) : y(x^2 + y^2 + axy + bxz + yz) : xyz)$$

with $a, b \in \mathbb{C}$, $a^2 \neq 4$ and $2b \notin \{a \pm \sqrt{a^2 - 4}\}$. The structure of the set of cubic birational maps is much more complicated ([5, Chapter 6]), nevertheless one has the following result.

Theorem 2.2 (Proposition 6.35, Theorem 6.38, [5]). — *The closure of*

$$\mathcal{X} = \{ \mathcal{O}(\Phi_{a,b}) \mid a, b \in \mathbb{C}, a^2 \neq 4, 2b \notin \{a \pm \sqrt{a^2 - 4}\} \}$$

in the set of rational maps of degree 3 is an irreducible algebraic variety of dimension 18.

Furthermore the closure of \mathcal{X} in $\mathring{\text{Bir}}_3$ is $\mathring{\text{Bir}}_3$.

The "most degenerate model" ⁽¹⁾ is up to automorphisms of $\mathbb{P}_{\mathbb{C}}^2$

$$\Psi: (x : y : z) \dashrightarrow (xz^2 + y^3 : yz^2 : z^3).$$

1. In the following sense: for any ϕ in Bir_3 the following inequality holds: $\dim \mathcal{O}(\phi) \geq \dim \mathcal{O}(\Psi) = 13$.

2.2. About foliations. —

Definition 2.3. — Let \mathcal{F} be a foliation (maybe singular) on a complex manifold M ; the foliation \mathcal{F} is a **singular transversely projective** one if there exists

- a) $\pi: P \rightarrow M$ a \mathbb{P}^1 -bundle over M ,
- b) \mathcal{G} a codimension one singular holomorphic foliation on P transversal to the generic fibers of π ,
- c) $\varsigma: M \rightarrow P$ a meromorphic section generically transverse to \mathcal{G} ,

such that $\mathcal{F} = \varsigma^* \mathcal{G}$.

Let \mathcal{F} be a foliation on $\mathbb{P}_{\mathbb{C}}^2$; assume that there exist three rational 1-forms θ_0, θ_1 and θ_2 on $\mathbb{P}_{\mathbb{C}}^2$ such that

- i) \mathcal{F} is described by θ_0 , i.e. $\mathcal{F} = \mathcal{F}_{\theta_0}$,
- ii) the θ_i 's form a $\mathfrak{sl}(2; \mathbb{C})$ -triplet, that is

$$d\theta_0 = \theta_0 \wedge \theta_1, \quad d\theta_1 = \theta_0 \wedge \theta_2, \quad d\theta_2 = \theta_1 \wedge \theta_2.$$

Then \mathcal{F} is a singular transversely projective foliation. To see it one considers the manifolds $M = \mathbb{P}_{\mathbb{C}}^2$, $P = \mathbb{P}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^1$, the canonical projection $\pi: P \rightarrow M$ and the foliation \mathcal{G} given by

$$\theta = dz + \theta_0 + z\theta_1 + \frac{z^2}{2}\theta_2$$

where z is an affine coordinate of $\mathbb{P}_{\mathbb{C}}^1$; in that case the transverse section is $z = 0$. When one can choose the θ_i 's such that $\theta_1 = \theta_2 = 0$ (resp. $\theta_2 = 0$) the foliation \mathcal{F} is **defined by a closed 1-form** (resp. is **transversely affine**).

Classical examples of singular transversely projective foliations are given by Riccati foliations.

Definition 2.4. — A **Riccati equation** is a differential equation of the following type

$$\mathcal{E}_R: y' = a(x)y^2 + b(x)y + c(x)$$

where a, b and c are meromorphic functions on an open subset \mathcal{U} of \mathbb{C} . To the equation \mathcal{E}_R one associates the meromorphic differential form

$$\omega_{\mathcal{E}_R} = dy - (a(x)y^2 + b(x)y + c(x)) dx$$

defined on $\mathcal{U} \times \mathbb{C}$. In fact $\omega_{\mathcal{E}_R}$ induces a foliation $\mathcal{F}_{\omega_{\mathcal{E}_R}}$ on $\mathcal{U} \times \mathbb{P}_{\mathbb{C}}^1$ that is transverse to the generic fiber of the projection $\mathcal{U} \times \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathcal{U}$. One can check that

$$\theta_0 = \omega_{\mathcal{E}_R}, \quad \theta_1 = -(2a(x)y + b(x)) dx, \quad \theta_2 = -2a(x) dx$$

is a $\mathfrak{sl}(2; \mathbb{C})$ -triplet associated to the foliation $\mathcal{F}_{\omega_{\mathcal{E}_R}}$.

We say that $\omega_{\mathcal{E}_R}$ is a **Riccati 1-form** and $\mathcal{F}_{\omega_{\mathcal{E}_R}}$ is a **Riccati foliation**.

Let S be a ruled surface, that is a surface S endowed with $f: S \rightarrow C$, where C denotes a curve and $f^{-1}(c) \simeq \mathbb{P}_{\mathbb{C}}^1$. Let us consider a singular foliation \mathcal{F} on S transverse to the generic fibers of f . The foliation \mathcal{F} is transversely projective.

Recall that a foliation \mathcal{F} is **radial** at a point m of the surface M if in local coordinates (x, y) around m the foliation \mathcal{F} is given by a holomorphic 1-form of the following type

$$\omega = x dy - y dx + \text{h.o.t.}$$

Let us denote by $\mathbb{F}(n; d)$ the set of foliations of degree d on $\mathbb{P}_{\mathbb{C}}^n$ (see [2]). The following statement gives a criterion which asserts that an element of $\mathbb{F}(2; 2)$ is transversely projective.

Proposition 2.5. — *Let $\mathcal{F} \in \mathbb{F}(2;2)$ be a foliation of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$. If a singular point of \mathcal{F} is radial, then \mathcal{F} is transversely projective.*

Proof. — Assume that the singular point is the origin 0 in the affine chart $z = 1$, the foliation \mathcal{F} is thus defined by a 1-form of the following type

$$\omega = xdy - ydx + q_1 dx + q_2 dy + q_3(xdy - ydx)$$

where the q_i 's denote quadratic forms. Let us consider the complex projective plane $\mathbb{P}_{\mathbb{C}}^2$ blown up at the origin; this space is denoted by $\text{Bl}(\mathbb{P}_{\mathbb{C}}^2, 0)$. Let $\pi: \text{Bl}(\mathbb{P}_{\mathbb{C}}^2, 0) \rightarrow \mathbb{P}_{\mathbb{C}}^2$ be the canonical projection. Then $\pi^*\mathcal{F}$ is transverse to the generic fibers of π , and in fact transverse to all the fibers excepted the strict transforms of the lines $xq_1 + yq_2 = 0$. Hence the foliation $\pi^*\mathcal{F}$ is transversely projective; since this notion is invariant under the action of a birational map, \mathcal{F} is transversely projective. \square

Remark 2.6. — The same argument can be involved for foliations of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$ having a singular point with zero 1-jet.

Remark 2.7. — The closure of the set Δ_R of foliations in $\mathbb{F}(2;2)$ having a radial singular point is irreducible, of codimension 2 in $\mathbb{F}(2;2)$.

3. Proof of Theorem A

We establish Theorem A in two steps: we first look at foliations that have at least two singular points and then at foliations with exactly one singular point.

3.1. Foliations of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$ with at least two singularities. —

Proposition 3.1. — *For any $\mathcal{F} \in \mathbb{F}(2;2)$ with at least two distinct singularities there exists a quadratic birational map $\psi \in \mathcal{O}(\rho)$ such that $\deg \psi^*\mathcal{F} \leq 3$.*

Proof. — In homogeneous coordinates \mathcal{F} is described by a 1-form

$$\omega = q_1 yz \left(\frac{dy}{y} - \frac{dz}{z} \right) + q_2 xz \left(\frac{dz}{z} - \frac{dx}{x} \right) + q_3 xy \left(\frac{dx}{x} - \frac{dy}{y} \right)$$

where

$$\begin{aligned} q_1 &= a_0x^2 + a_1y^2 + a_2z^2 + a_3xy + a_4xz + a_5yz, & q_2 &= b_0x^2 + b_1y^2 + b_2z^2 + b_3xy + b_4xz + b_5yz, \\ q_3 &= c_0x^2 + c_1y^2 + c_2z^2 + c_3xy + c_4xz + c_5yz. \end{aligned}$$

Up to an automorphism of $\mathbb{P}_{\mathbb{C}}^2$ one can suppose that $(1:0:0)$ and $(0:1:0)$ are singular points of \mathcal{F} , that is $a_1 = b_0 = c_0 = c_1 = 0$. If $c_3 \neq 0$, resp. $c_3 = 0$ and $b_4 \neq 0$, resp. $c_3 = b_4 = 0$, then let us consider the quadratic birational map ψ of $\mathcal{O}(\rho)$ defined as follows

$$\psi: (x:y:z) \dashrightarrow \left(xy: z^2 + \frac{b_3 - c_4 + \sqrt{(b_3 - c_4)^2 + 4b_4c_3}}{2c_3} yz : yz \right),$$

resp.

$$\psi: (x:y:z) \dashrightarrow \left(xy: z^2 + yz: -\frac{b_3 - c_4}{b_4} yz \right),$$

resp. $\psi = \rho$. A direct computation shows that $\psi^*\omega = yz^2\omega'$ where ω' denotes a homogeneous 1-form of degree 4. The foliation \mathcal{F}' defined by ω' has degree at most 3. \square

3.2. Foliations of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$ with exactly one singularity. — Such foliations have been classified:

Theorem 3.2 ([6]). — *Up to automorphisms of $\mathbb{P}_{\mathbb{C}}^2$ there are four foliations of degree 2 on $\mathbb{P}_{\mathbb{C}}^2$ having exactly one singularity. They are described in affine chart by the following 1-forms:*

- $\Omega_1 = x^2 dx + y^2(x dy - y dx)$,
- $\Omega_2 = x^2 dx + (x + y^2)(x dy - y dx)$,
- $\Omega_3 = xy dx + (x^2 + y^2)(x dy - y dx)$,
- $\Omega_4 = (x + y^2 - x^2 y) dy + x(x + y^2) dx$.

Proposition 3.3. — *There exists a quadratic birational map $\psi_1 \in \mathcal{O}(\rho)$ such that $\deg \psi_1^* \mathcal{F}_{\Omega_1} = 2$; furthermore \mathcal{F}_{Ω_1} has a rational first integral and is non-primitive.*

For $k = 2, 3$, there is no birational map ϕ_k such that $\deg \phi_k^ \mathcal{F}_{\Omega_k} = 0$ but there is a $\psi_k \in \mathcal{O}(\tau)$ such that $\deg \psi_k^* \mathcal{F}_{\Omega_k} = 1$. In particular \mathcal{F}_{Ω_2} and \mathcal{F}_{Ω_3} are non-primitive.*

There is a quadratic birational map $\psi_4 \in \mathcal{O}(\tau)$ such that $\deg \psi_4^ \mathcal{F}_{\Omega_4} = 3$ and \mathcal{F}_{Ω_4} is primitive.*

Remark 3.4. — If $\phi = (x^2 : xy : xz + y^2)$, then $\deg \phi^* \mathcal{F}_{\Omega_2} = \deg \phi^* \mathcal{F}_{\Omega_3} = 2$. A contrario we will see later there is no quadratic birational map ϕ such that $\deg \phi^* \mathcal{F}_{\Omega_4} = 2$ (see Corollary 4.15).

Corollary 3.5. — *For any element \mathcal{F} of $\mathbb{F}(2; 2)$ with exactly one singularity there exists a quadratic birational map ψ such that $\deg \psi^* \mathcal{F} \leq 3$.*

Proof of Proposition 3.3. — The foliation \mathcal{F}_{Ω_1} is given in homogeneous coordinates by

$$\Omega'_1 = (x^2 z - y^3) dx + xy^2 dy - x^3 dz;$$

if $\psi_1 : (x : y : z) \dashrightarrow (x^2 : xy : yz)$ then

$$\psi_1^* \Omega'_1 \wedge (y(2xz - y^2) dx + x(y^2 - xz) dy - x^2 y dz) = 0.$$

The foliation \mathcal{F}_{Ω_1} has a rational first integral and is non-primitive, it is the image of a foliation of degree 0 by a cubic birational map:

$$(x^3 : x^2 y : x^2 z + y^3/3)^* \Omega'_1 \wedge (z dx - x dz) = 0.$$

The foliation \mathcal{F}_{Ω_2} is described in homogeneous coordinates by

$$\Omega'_2 = (x^2 z - xyz - y^3) dx + x(xz + y^2) dy - x^3 dz;$$

let us consider the birational map $\psi_2 : (x : y : z) \dashrightarrow (x^2 : xy : xz - 2x^2 - 2xy - y^2)$ then

$$\psi_2^* \Omega'_2 \wedge ((xz - yz) dx + xz dy - x^2 dz) = 0.$$

One can verify that

$$\left(2 + \frac{1}{x} + 2\frac{y}{x} + \frac{y^2}{x^2}\right) \exp\left(-\frac{y}{x}\right)$$

is a first integral of \mathcal{F}_{Ω_2} ; it is easy to see that \mathcal{F}_{Ω_2} has no rational first integral so there is no birational map ϕ_2 such that $\deg \phi_2^* \mathcal{F}_{\Omega_2} = 0$.

The foliation \mathcal{F}_{Ω_3} is given in homogeneous coordinates by the 1-form

$$\Omega'_3 = y(xz - x^2 - y^2) dx + x(x^2 + y^2) dy - x^2 y dz;$$

if $\psi_3 : (x : y : z) \dashrightarrow (x^2 : xy : xz + y^2/2)$ then

$$\psi_3^* \Omega'_3 \wedge (y(z - x) dx + x^2 dy - xy dz) = 0.$$

The function

$$\left(\frac{y}{x}\right) \exp\left(\frac{1}{2} \frac{y^2}{x^2} - \frac{1}{x}\right)$$

is a first integral of \mathcal{F}_{Ω_3} and \mathcal{F}_{Ω_3} has no rational first integral so there is no birational map φ_3 such that $\deg \varphi_3^* \mathcal{F}_{\Omega_3} = 0$.

Let us consider the birational map of $\mathbb{P}_{\mathbb{C}}^2$ given by

$$\Psi_4: (x : y : z) \dashrightarrow (-x^2 : xy : y^2 - xz)$$

In homogeneous coordinates $\Omega'_4 = x(xz + y^2) dx + (xz^2 + y^2z - x^2y) dy + (xyz - y^3 - x^3) dz$; a direct computation shows that

$$\Psi_4^* \Omega'_4 \wedge ((3y^3z - x^2y^2 + x^3z - 2xyz^2) dx + (x^3y - 4y^4 - x^2z^2 + 3xy^2z) dy + x(2y^3 - x^3 - xyz) dz) = 0.$$

The foliation \mathcal{F}_{Ω_4} has no invariant algebraic curve so \mathcal{F}_{Ω_4} is not transversely projective ([6, Proposition 1.3]). In fact a foliation of degree 2 without invariant algebraic curve is primitive; as a consequence \mathcal{F}_{Ω_4} is a primitive foliation. \square

4. Numerical invariance

In the sequel num. inv. means numerically invariant.

In this section we determine the foliations \mathcal{F} of $\mathbb{F}(2;2)$ num. inv. under the action of σ (resp. ρ , resp. τ). Note that if ϕ is a birational map of $\mathbb{P}_{\mathbb{C}}^2$ and ℓ an element of $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$ then $\deg(\phi\ell)^* \mathcal{F} = \deg \phi^* \mathcal{F}$; hence following Theorem 2.1 we get the description of foliations num. inv. under the action of a quadratic birational map of $\mathbb{P}_{\mathbb{C}}^2$.

Lemma 4.1. — *An element \mathcal{F} of $\mathbb{F}(2;2)$ is num. inv. under the action of σ if and only if it is given up to permutations of coordinates and standard affine charts by 1-forms of the following type*

- either $\omega_1 = y(\kappa + \varepsilon y) dx + (\beta x + \delta y + \alpha x^2 + \gamma xy) dy$,
- or $\omega_2 = (\delta + \beta y + \kappa y^2) dx + (\alpha + \varepsilon x + \gamma x^2) dy$,

where $\alpha, \beta, \gamma, \delta, \varepsilon, \kappa$ (resp. $\alpha, \beta, \gamma, \delta, \varepsilon, \kappa$) are complex numbers such that $\deg \mathcal{F}_{\omega_1} = 2$ (resp. $\deg \mathcal{F}_{\omega_2} = 2$).

Proof. — The foliation \mathcal{F} is defined by a homogeneous 1-form ω of degree 3. The map σ is an automorphism of $\mathbb{P}_{\mathbb{C}}^2 \setminus \{xyz = 0\}$ so if $\sigma^* \omega = P\omega'$, with ω' a 1-form of degree 3 and P a homogeneous polynomial then $P = x^i y^j z^k$ for some integers i, j, k such that $i + j + k = 4$. Up to permutation of coordinates it is sufficient to look at the four following cases: $P = x^4$, $P = x^3 y$, $P = x^2 y^2$ and $P = x^2 y z$. Let us write ω as follows

$$\omega = q_1 y z \left(\frac{dy}{y} - \frac{dz}{z} \right) + q_2 x z \left(\frac{dz}{z} - \frac{dx}{x} \right) + q_3 x y \left(\frac{dx}{x} - \frac{dy}{y} \right)$$

where

$$\begin{aligned} q_1 &= a_0 x^2 + a_1 y^2 + a_2 z^2 + a_3 xy + a_4 xz + a_5 yz, & q_2 &= b_0 x^2 + b_1 y^2 + b_2 z^2 + b_3 xy + b_4 xz + b_5 yz, \\ q_3 &= c_0 x^2 + c_1 y^2 + c_2 z^2 + c_3 xy + c_4 xz + c_5 yz. \end{aligned}$$

Computations show that x^4 (resp. $x^3 y$) cannot divide $\sigma^* \omega$. If $P = x^2 y z$ then $\sigma^* \omega = P\omega'$ if and only if

$$c_0 = 0, \quad b_0 = 0, \quad a_2 = 0, \quad b_2 = 0, \quad a_1 = 0, \quad c_1 = 0, \quad b_4 = 0, \quad c_3 = 0, \quad b_3 = c_4$$

that gives ω_1 . Finally one has $\sigma^* \omega = x^2 y^2 \omega'$ if and only if

$$c_1 = 0, \quad c_0 = 0, \quad b_0 = 0, \quad a_1 = 0, \quad b_4 = 0, \quad c_3 = 0, \quad a_5 = 0, \quad b_3 = c_4, \quad c_5 = a_3;$$

in that case we obtain ω_2 . \square

Proposition 4.2. — *A foliation $\mathcal{F} \in \mathbb{F}(2;2)$ num. inv. under the action of an element of $\mathcal{O}(\sigma)$ is $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$ -conjugate either to a foliation of type \mathcal{F}_{ω_1} , or to a foliation of type \mathcal{F}_{ω_2} ; in particular it is transversely projective.*

Proof. — Let ϕ be an element of $\mathcal{O}(\sigma)$ such that $\deg \phi^* \mathcal{F} = 2$; the map ϕ can be written $\ell_1 \sigma \ell_2$ where ℓ_1 and ℓ_2 denote automorphisms of $\mathbb{P}_{\mathbb{C}}^2$. By assumption the degree of $(\ell_1 \sigma \ell_2)^* \mathcal{F} = \ell_2^*(\sigma^*(\ell_1^* \mathcal{F}))$ is 2. Hence $\deg \sigma^*(\ell_1^* \mathcal{F}) = 2$ and the foliation $\ell_1^* \mathcal{F}$ is num. inv. under the action of σ . Since $\ell_1^* \mathcal{F}$ and \mathcal{F} are conjugate and since the notion of transversal projectivity is invariant by conjugacy it is sufficient to establish the statement for $\phi = \sigma$. The proposition thus follows from the fact that 1-forms of Lemma 4.1 are Riccati ones (up to multiplication). \square

Remark 4.3. — For generic values of parameters $\alpha, \beta, \gamma, \delta, \varepsilon, \kappa$ a foliation of type \mathcal{F}_{ω_1} given by the corresponding form ω_1 is not given by a closed meromorphic 1-form. This can be seen by studying the holonomy group of \mathcal{F}_{ω_1} that can be identified with a subgroup of $\text{PGL}(2; \mathbb{C})$ generated by two elements f and g . For generic values of the parameters f and g are also generic, in particular the group $\langle f, g \rangle$ is free. When \mathcal{F}_{ω_1} is given by a closed 1-form, then the holonomy group is an abelian one.

Remark that a contrario the foliations given by 1-forms of type ω_2 are given by a closed meromorphic 1-form.

Remark 4.4. — Let Δ_i denote the closure of the set of elements of $\mathbb{F}(2;2)$ conjugate to a foliation of type \mathcal{F}_{ω_i} . The following inclusion holds: $\Delta_2 \subset \Delta_1$.

Note also that Δ_1 is contained in Δ_R (see Remark 2.7).

Remark 4.5. — The notion of num. inv. is not related to the dynamic of the map (see [3] for example): the foliations num. inv. by the involution σ ("without dynamic") are conjugate to the foliations num. inv. by $A\sigma$, $A \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$, which has a rich dynamic for generic A .

The foliations of $\mathbb{F}(2;2)$ invariant by σ are particular cases of num. inv. foliations:

Proposition 4.6. — *An element of $\mathbb{F}(2;2)$ invariant by σ is given up to permutations of coordinates and affine charts*

- either by $y(1+y)dx + (\beta x + \alpha y + \alpha x^2 + \beta xy)dy$,
- or by $y(1-y)dx + (\beta x - \alpha y + \alpha x^2 - \beta xy)dy$,
- or by $ydx + (\alpha + \varepsilon x + \alpha x^2)dy$,

where the parameters are complex numbers such that the degree of the associated foliations is 2.

Proof. — With the notations of Lemma 4.1 one has

$$\sigma^* \omega_1 = -y(\varepsilon + \kappa y)dx - (\gamma x + \alpha y + \delta x^2 + \beta xy)dy;$$

thus $\sigma^* \omega_1 \wedge \omega_1 = 0$ if and only if either $\gamma = \beta$, $\delta = \alpha$, $\varepsilon = \kappa$, or $\gamma = -\beta$, $\delta = -\alpha$, $\varepsilon = -\kappa$.

One has $\sigma^* \omega_2 = -(\kappa + \beta y + \delta y^2)dx - (\gamma + \varepsilon x + \alpha x^2)dy$ and $\omega_2 \wedge \sigma^* \omega_2 = 0$ if and only if $\gamma = \alpha$, $\delta = 0$ and $\kappa = 0$. \square

Remark 4.7. — The foliations associated to the two first 1-forms with parameters α, β of Proposition 4.6 are conjugate by the automorphism $(x, y) \mapsto (x, -y)$.

Lemma 4.8. — *A foliation $\mathcal{F} \in \mathbb{F}(2;2)$ is num. inv. under the action of ρ if and only if \mathcal{F} is given in affine chart*

- either by $\omega_3 = y(\kappa + \epsilon y + \lambda y^2) dx + (\beta + \kappa x + \delta y + \gamma xy + \alpha y^2 - \lambda xy^2) dy$,
- or by $\omega_4 = y(\mu + \delta x + \gamma y + \epsilon xy) dx + (\alpha + \beta x + \lambda y + \delta x^2 + \kappa xy - \epsilon x^2 y) dy$,
- or by $\omega_5 = (\lambda + \gamma y + \kappa xy + \epsilon y^2) dx + (\beta + \delta x + \alpha x^2) dy$,

where the parameters are such that the degree of the corresponding foliations is 2.

Proof. — Let us take the notations of the proof of Lemma 4.1. The map ρ is an automorphism of $\mathbb{P}_{\mathbb{C}}^2 \setminus \{yz = 0\}$ so if $\rho^*\omega = P\omega'$ with ω' a 1-form of degree 3 and P a homogeneous polynomial then $P = y^j z^k$ for some integer j, k such that $j + k = 4$. We have to look at the four following cases: $P = z^4$, $P = yz^3$, $P = y^2 z^2$, $P = y^3 z$ and $P = y^4$. Computations show that y^4 (resp. $y^3 z$) cannot divide $\rho^*\omega$. If $P = z^4$ then $\rho^*\omega = P\omega'$ if and only if

$$c_0 = 0, \quad b_0 = 0, \quad c_3 = 0, \quad b_4 = 0, \quad b_2 = 0, \quad a_0 = c_4, \quad b_3 = c_4, \quad a_4 = 2c_2 - b_5;$$

this gives the first case ω_3 . The equality $\rho^*\omega = yz^3\omega'$ holds if and only if

$$b_0 = 0, \quad c_0 = 0, \quad b_4 = 0, \quad c_1 = 0, \quad a_1 = 0, \quad b_2 = 0, \quad a_0 = 2c_4 - b_3$$

and we obtain ω_4 . Finally one has $\rho^*\omega = y^2 z^2 \omega'$ if and only if

$$c_1 = 0, \quad b_0 = 0, \quad c_3 = 0, \quad a_5 = 0, \quad a_1 = 0, \quad c_0 = 0, \quad b_4 = 0, \quad c_5 = a_3$$

which corresponds to ω_5 . □

Proposition 4.9. — *The foliations of type \mathcal{F}_{ω_3} and \mathcal{F}_{ω_5} are transversely projective. In fact the \mathcal{F}_{ω_3} are transversely affine and the \mathcal{F}_{ω_5} are Riccati ones.*

Proof. — A foliation of type \mathcal{F}_{ω_3} is described by the 1-form

$$\theta_0 = dx - \frac{(\beta + \delta y + \alpha y^2) + (\kappa + \gamma y - \lambda y^2)x}{y(\kappa + \epsilon y + \lambda y^2)} dy$$

and it is transversely affine; to see it consider the $\mathfrak{sl}(2; \mathbb{C})$ -triplet

$$\theta_0, \quad \theta_1 = \frac{\kappa + \gamma y - \lambda y^2}{y(\kappa + \epsilon y + \lambda y^2)} dy, \quad \theta_2 = 0.$$

A foliation of type \mathcal{F}_{ω_5} is given by

$$dy + \frac{\lambda + (\gamma + \kappa)y + \epsilon y^2}{\beta + \delta x + \alpha x^2} dx$$

and thus is a Riccati foliation. In fact the fibration $x/z = \text{constant}$ is transverse to \mathcal{F}_{ω_5} that generically has three invariant lines. □

We don't know if the \mathcal{F}_{ω_4} are transversely projective. For generic values of the parameters a foliation of type \mathcal{F}_{ω_4} hasn't meromorphic uniform first integral in the affine chart $z = 1$. Thus if \mathcal{F}_{ω_4} is transversely projective then it must have an invariant algebraic curve different from $z = 0$ (see [7]). We don't know if it is the case. A foliation of degree 2 is conjugate to a generic \mathcal{F}_{ω_4} (by an automorphism of $\mathbb{P}_{\mathbb{C}}^2$) if and only if it has an invariant line (say $y = 0$) with a singular point (say 0) and local model $2xdy - ydx$. The closure of the set of such foliations has codimension 2. Note that the three families \mathcal{F}_{ω_3} , \mathcal{F}_{ω_4} and \mathcal{F}_{ω_5} have non trivial intersection. The set $\overline{\{\mathcal{F}_{\omega_4}\}}$ contains many interesting elements such that the famous Euler foliation given by $y^2 dx + (y - x) dy$; this foliation is transversely affine but is not given by a closed rational 1-form.

Proposition 4.10. — *A foliation $\mathcal{F} \in \mathbb{F}(2; 2)$ num. inv. under the action of an element of $\mathcal{O}(\rho)$ is conjugate to a foliation either of type \mathcal{F}_{ω_3} , or of type \mathcal{F}_{ω_4} , or of type \mathcal{F}_{ω_5} .*

Let us look at special num. inv. foliations, those invariant by ρ .

Proposition 4.11. — An element of $\mathbb{F}(2;2)$ invariant by ρ is given by a 1-form of one of the following type

- $y(1-y)dx + (\beta+x)dy$,
- $y^2dx + (-1+y)dy$,
- $y(1-y)(\gamma+\delta x)dx + (1+y)(\alpha+\beta x+\delta x^2)dy$,
- $y(1+y)(\gamma+\delta x)dx + (1-y)(\alpha+\beta x+\delta x^2)dy$,
- $(1-y^2)dx + (\beta+\delta x+\alpha x^2)dy$,

where the parameters are complex numbers such that the degree of the associated foliations is 2.

Corollary 4.12. — An element of $\mathbb{F}(2;2)$ invariant by ρ is defined by a closed 1-form.

Remark 4.13. — The third and fourth cases with parameters $\alpha, \beta, \gamma, \delta$ are conjugate by the automorphism $(x, y) \mapsto (x, -y)$.

From Lemmas 4.1 and 4.8 one gets the following statement.

Proposition 4.14. — A foliation num. inv. by an element of $\mathcal{O}(\phi)$, with $\phi = \sigma, \rho$, preserves an algebraic curve.

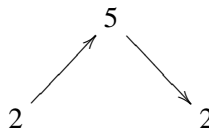
Corollary 4.15. — There is no quadratic birational map ϕ of $\mathbb{P}_{\mathbb{C}}^2$ such that $\deg \phi^* \mathcal{F}_{\Omega_4} = 2$.

Proof. — The foliation \mathcal{F}_{Ω_4} has no invariant algebraic curve ([6, Proposition 1.3]); according to Proposition 4.14 it is thus sufficient to show that there is no birational map $\phi \in \mathcal{O}(\tau)$ such that $\deg \phi^* \mathcal{F}_{\Omega_4} = 2$ that can be established with a direct and tedious computation. \square

Remark 4.16. — The map ρ can be written $\ell_1 \sigma \ell_2 \sigma \ell_3$ with

$$\ell_1 = (z-y : y-x : y), \quad \ell_2 = (y+z : z : x), \quad \ell_3 = (x+z : y-z : z).$$

We are interested by the "intermediate" degrees of a numerically invariant foliation \mathcal{F} , that is the sequence $\deg \mathcal{F}, \deg(\ell_1 \sigma)^* \mathcal{F}, \deg(\ell_1 \sigma \ell_2 \sigma \ell_3)^* \mathcal{F} = \deg \mathcal{F}$. A tedious computation shows that for generic values of the parameters the sequence is 2, 5, 2. We schematize this fact by the diagram



A similar argument to Lemma 4.1 yields to the following result.

Lemma 4.17. — An element \mathcal{F} of $\mathbb{F}(2;2)$ is num. inv. under the action of τ if and only if \mathcal{F} is given in affine chart by a 1-form of type

$$\begin{aligned} \omega_6 = & (-\delta x + \alpha y - \varepsilon x^2 + \theta xy + \beta y^2 + \kappa x^2 y + \mu xy^2 + \lambda y^3) dx \\ & + (-3\alpha x + \xi x^2 + 2(\delta - \beta)xy + \alpha y^2 - \kappa x^3 - \mu x^2 y - \lambda xy^2) dy \end{aligned}$$

where the parameters are such that $\deg \mathcal{F}_{\omega_6} = 2$.

We don't know the qualitative description of foliations of type \mathcal{F}_{ω_6} . For example we don't know if the \mathcal{F}_{ω_6} are transversely projective. If it is the case, this implies the existence of invariant algebraic curves, and that fact is unknown.

Proposition 4.18. — A foliation $\mathcal{F} \in \mathbb{F}(2;2)$ num. inv. under the action of an element of $\mathcal{O}(\tau)$ is conjugate to \mathcal{F}_{ω_6} for suitable values of the parameters.

Let us describe some special num. inv. foliations under the action of τ , those invariant by τ .

Proposition 4.19. — An element of $\mathbb{F}(2;2)$ invariant by τ is given

- either by $(-\epsilon x^2 + \theta xy + \beta y^2 + \epsilon xy^2 - (\frac{\xi}{2} + \theta)y^3) dx + x(\xi x - 2\beta y - \epsilon xy + (\frac{\xi}{2} + \theta)y^2) dy$,
- or by $(-\delta x + \alpha y + \frac{3}{2}\delta y^2 + \kappa x^2 y + \mu xy^2 + \lambda y^3) dx - (3\alpha x + \delta xy - \alpha y^2 + \kappa x^3 + \mu x^2 y + \lambda xy^2) dy$,

where the parameters are complex numbers such that the degree of the associated foliations is 2.

The foliations associated to the first 1-form are transversely affine.

Proof. — The 1-jet at the origin of the 1-form

$$\omega = (-\epsilon x^2 + \theta xy + \beta y^2 + \epsilon xy^2 - (\frac{\xi}{2} + \theta)y^3) dx + x(\xi x - 2\beta y - \epsilon xy + (\frac{\xi}{2} + \theta)y^2) dy$$

is zero so after one blow-up \mathcal{F}_{ω} is transverse to the generic fiber of the Hopf fibration; furthermore as the exceptional divisor is invariant, \mathcal{F}_{ω} is transversely affine. \square

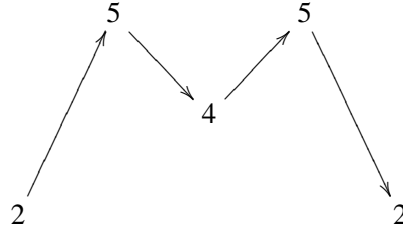
Remark 4.20. — The map τ can be written $\ell_1 \sigma \ell_2 \sigma \ell_3 \sigma \ell_2 \sigma \ell_4$ with

$$\begin{aligned} \ell_1 &= (x - y : x - 2y : -x + y - z), & \ell_2 &= (x + z : x : y), \\ \ell_3 &= (-y : x - 3y + z : x), & \ell_4 &= (y - x : z - 2x : 2x - y). \end{aligned}$$

Let us consider a foliation \mathcal{F} num. inv. under the action of τ ; set $\mathcal{F}' = \ell_1^* \mathcal{F}$. We compute the intermediate degrees:

$$\deg \sigma^* \mathcal{F}' = 5, \quad \deg(\sigma \ell_2 \sigma)^* \mathcal{F}' = 4, \quad \deg(\sigma \ell_3 \sigma \ell_2 \sigma)^* \mathcal{F}' = 5.$$

To summarize:



5. Higher degree

We will now focus on similar questions but with cubic birational maps of $\mathbb{P}_{\mathbb{C}}^2$ and elements of $\mathbb{F}(2;2)$. The generic model of such birational maps is:

$$\Phi_{a,b}: (x : y : z) \dashrightarrow (x(x^2 + y^2 + axy + bxz + yz) : y(x^2 + y^2 + axy + bxz + yz) : xyz)$$

with $a, b \in \mathbb{C}$, $a^2 \neq 4$ and $2b \notin \{a \pm \sqrt{a^2 - 4}\}$.

Lemma 5.1. — An element \mathcal{F} of $\mathbb{F}(2;2)$ is num. inv. under the action of $\Phi_{a,b}$ if and only if \mathcal{F} is given in affine chart

- either by $\omega_7 = y(\alpha + \gamma) dx - x(\alpha + \kappa x) dy$,
- or by $\omega_8 = b(b^2 - ab + 1 + (a - 2b)y + y^2) dx + ((b^2 - ab + 1) + (ab - 2)x + x^2) dy$,

where the parameters are such that $\deg \mathcal{F}_{\omega_7} = \deg \mathcal{F}_{\omega_8} = 2$.

Remark 5.2. — Remark that the foliations \mathcal{F}_{ω_7} do not depend on the parameters of $\Phi_{a,b}$, that is, the \mathcal{F}_{ω_7} are num. inv. by all $\Phi_{a,b}$, whereas the \mathcal{F}_{ω_8} only depend on a and b .

Furthermore \mathcal{F}_{ω_7} is num. inv. by σ and ρ .

Proposition 5.3. — Any $\mathcal{F} \in \mathbb{F}(2;2)$ num. inv. under the action of $\Phi_{a,b}$, and more generally any $\mathcal{F} \in \mathbb{F}(2;2)$ num. inv. under the action of a generic cubic birational map of $\mathbb{P}_{\mathbb{C}}^2$, satisfies the following properties:

- \mathcal{F} is given by a rational closed 1-form;
- \mathcal{F} is non-primitive.

Proof. — Let us establish those properties for \mathcal{F}_{ω_7} .

For generic values of α, γ and κ one can assume up to linear conjugacy that \mathcal{F}_{ω_7} is given by

$$\eta' = y(1+y)dx - x(1+x)dy$$

that gives up to multiplication

$$\frac{dx}{x(1+x)} - \frac{dy}{y(1+y)}$$

which is closed. A foliation of type \mathcal{F}_{ω_7} is also described in homogeneous coordinates by the 1-form

$$\eta = yz(y+z)dx - xz(x+z)dy + xy(x-y)dz.$$

One has

$$\sigma^*\eta = xyz(-(y+z)dx + (x+z)dy + (x-y)dz)$$

so \mathcal{F}_{ω_7} is non-primitive.

The idea and result are the same for the foliations \mathcal{F}_{ω_8} (except that it gives a birational map ϕ such that $\deg \phi^* \mathcal{F}_{\omega_8} = 1$). \square

Let us consider an element \mathcal{F} of $\mathbb{F}(2;2)$ num. inv. under the action of a birational map of degree ≥ 3 ; is \mathcal{F} defined by a closed 1-form ?

Remark 5.4. — The foliations \mathcal{F}_{ω_7} are contained in the orbit of the foliation $\mathcal{F}_{\eta'}$.

Remark 5.5. — Any map $\Phi_{a,b}$ can be written $\ell_1 \sigma \ell_2 \sigma \ell_3$ with $\ell_2 = (*y + *z : *y + *z : *x + *y + *z)$ (see [5, Proposition 6.36]). Let us consider the birational map $\xi = \sigma \ell_2 \sigma$ with

$$\ell_2 = (ay + bz : cy + ez : fx + gy + hz) \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^2).$$

As in Lemma 5.1 there are two families of foliations $\mathcal{F}_1, \mathcal{F}_2$ of degree 2, one that does not depend on the parameters of ξ and the other one depending only on the parameters of ξ , such that $\xi^* \mathcal{F}_1$ and $\xi^* \mathcal{F}_2$ are of degree 2. One question is the following: what is the intermediate degree ? A computation shows that for generic parameters $\deg \sigma^* \mathcal{F}_1 = 4$ and that $\deg \sigma^* \mathcal{F}_2 = 2$. This implies in particular that \mathcal{F}_{ω_8} is num. inv. under the action of σ . For \mathcal{F}_1 and \mathcal{F}_{ω_7} one has

$$\begin{array}{ccc} & 4 & \\ \nearrow & & \searrow \\ 2 & & 2 \end{array}$$

and for \mathcal{F}_2 and \mathcal{F}_{ω_8}

$$2 \longrightarrow 2 \longrightarrow 2$$

Let us now consider the "most degenerate" cubic birational map

$$\Psi: (x : y : z) \dashrightarrow (xz^2 + y^3 : yz^2 : z^3).$$

Lemma 5.6. — An element \mathcal{F} of $\mathbb{F}(2;2)$ is num. inv. under the action of Ψ if and only if \mathcal{F} is given in affine chart by

$$\omega_9 = (-\alpha + \beta y + \gamma y^2) dx + (\varepsilon - 3\beta x + \kappa y - 3\gamma xy + \lambda y^2) dy$$

where the parameters are such that $\deg \mathcal{F}_{\omega_9} = 2$. In particular \mathcal{F} is transversely affine.

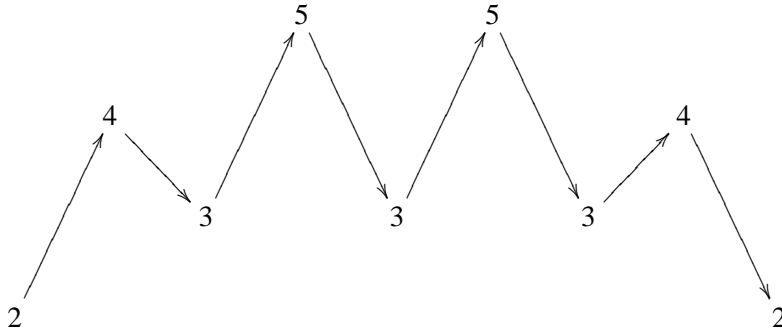
Remark 5.7. — The map ψ can be written $\ell_1 \sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma \ell_7$ with

$$\begin{aligned} \ell_1 &= (z - y : y : y - x), & \ell_2 &= (y + z : z : x), & \ell_3 &= (-z : -y : x - y), \\ \ell_4 &= (x + z : x : y), & \ell_5 &= (-y : x - 3y + z : x), & \ell_6 &= (-x : -y - z : x + y), \\ \ell_7 &= (x + y : z - y : y). \end{aligned}$$

As previously we consider the problem of the intermediate degrees; if $\mathcal{F}' = \ell_1^* \mathcal{F}$, a computation shows that for generic parameters

$$\begin{aligned} \deg \sigma^* \mathcal{F}' &= 4, & \deg(\sigma \ell_2 \sigma)^* \mathcal{F}' &= 3, & \deg(\sigma \ell_2 \sigma \ell_3 \sigma)^* \mathcal{F}' &= 5, \\ \deg(\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma)^* \mathcal{F}' &= 3, & \deg(\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma)^* \mathcal{F}' &= 5, \\ \deg(\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma)^* \mathcal{F}' &= 3, & \deg(\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma \ell_7)^* \mathcal{F}' &= 4, \end{aligned}$$

that is



We have not studied the quadratic foliations numerically invariant by (any) cubic birational transformation. It is reasonable to think that such foliations are transversely projective.

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